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LIMIT LAWS FOR SUMS OF PRODUCTS OF EXPONENTIALS OF *iid* RANDOM VARIABLES

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ABSTRACT

We study limit laws for sums of products of exponentials of nonnegative, *iid* random variables $\{V_{ij}\}$, namely $\sum_{i=1}^{N(n)} e^{\beta \sum_{j=1}^{n} V_{ij}}$. Under a Cramér type condition, $E[e^{sV_{ij}}] < \infty$ for some $s > 0$, a weak law of large numbers, central limit theorem, and convergence to stable laws is established for appropriate rates of growth of $N(n)$ and proper normalizations and scalings.

1. Introduction

The present work has several motivations. Our primary interest is to eventually gain an understanding of the dynamo problem in random flows. This is related to the study of magnetic fields generated by a conducting fluid in stars. For this model let $v(t, x)$ be a random, incompressible, velocity field on $\mathbf{R}^3, \kappa$ a small parameter (the inverse Reynolds number) and Δ the Laplacian on $\mathbf{R}^3$. Then

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the magnetic field H is the solution of the multi-component parabolic Anderson model,

$$\frac{\partial H}{\partial t} + < v, \nabla > H = \kappa \Delta H + < H, \nabla > v,$$
$$H(0,x) = H_0(x).$$

This is just Maxwell's equation. The field H has a multiplicative integral expression as an expectation over random paths satisfying

$$d\xi(s) = \sqrt{2\kappa} dw(s) - v(t-s, \xi(s))ds, \quad 0 \le s \le t,$$

and with a matrix potential $V = (V_{ij}) = (\partial v_i / \partial x_j)$ given by

$$H(t,x) = E_x\left[\prod_{s=0}^{t} (\delta_{ij} - V_{ij}(t-s, \xi(s)ds)) H_0(\xi(t))\right].$$

For details, see Molchanov and Ruzmaikin [5]. In a discrete approximation to this model one has

$$H(n,x) = E_x\left[\prod_{i=0}^{n} e^{C_{x_{n-i},i}} H_0(x_n)\right],$$

where C_{ij}, $0 \le j$, $i \in \mathbf{Z}^\mathbf{d}$ are *iid* trace zero random matrices and $\{x_n\}_{n \ge 0}$ is a Markov chain on $\mathbf{Z}^\mathbf{d}$ with transition probabilities

$$p_{ij} = \kappa, \quad |i-j| = 1, \qquad p_{ii} = 1 - 2d\kappa, \tag{1.1}$$

for some $0 < \kappa < 1/2d$.

A similar expression occurs in the study of phase transitions of random directed polymers on trees, as in Carpentier and Le Dousal [2] or Derrida and Spohn [3]. Here one considers a rooted binary tree with independent random energies attached to each branch. Denote by ϵ_{ij} the energy on branch j in generation i. For a length n path $\gamma = \{(x_0, 0), (x_1, 1), \ldots, (x_n, n)\}$ from the root let $V(\gamma) = \sum \epsilon_{x_i, i}$ be the sum of the energies along the path. The resulting partition function has the form $Z = \sum_\gamma e^{-\beta V(\gamma)}$, which is a multiple of the expected value of $e^{-\beta V(\gamma)}$ over random walk paths.

Another motivation is that of a model of a randomly moving particle on the integer lattice $\mathbb{Z}^d$ which encounters soft traps. The traps are modeled as *iid* random variables $\{V_{ij} : i \in \mathbb{Z}^d, j = 0, 1, 2, \ldots\}$. The particle follows a random walk on $\mathbb{Z}^d$ with transition probabilities as at (1.1).

Then the probability of survival up to time n of the walk in the environment $\{V_{ij}\}$ is given by

$$P_n(\gamma) = e^{-\beta \sum_{j=0}^{n} V_{x_j j}}$$

where γ is a realization of the Markov chain, $\gamma = \{(x_0, 0), (x_1, 1), (x_2, 2), \ldots\}$. Here $\beta > 0$ is some parameter. Let us write $(\Omega_m, \mathbb{P}_m(dw_m))$ as the probability space for the random media provided by the random variables $\{V_{ij}\}$. For each $w_m \in \Omega_m$, the random dynamics of the randomly moving particle among this field of 'soft' traps is represented by the probability space $(\Omega, \mathcal{F}^{w_m}, P^{w_m}(dw))$. If T is the killing time of the particle moving in the field provided by w_m then

$$P^{w_m}(T > n) = E^{w_m} e^{-\beta \sum_{j=0}^{n} V_{x_j j}}.$$

Consider now an initial configuration of particles concentrated on a box $Q_L = [-L, L]^d$,

$$\eta(0, i) = \begin{cases} 1, & i \in Q_L, \\ 0, & i \notin Q_L. \end{cases}$$

All particles move independently according to the dynamic described at (1.1).

Define

$$\eta(n, i, k) = \begin{cases} 1, & \text{if particle starting at } (0, i) \text{ survives to time } n \text{ and } x(n) = k, \\ 0, & \text{otherwise,} \end{cases}$$

and set

$$\eta(n, i) = \sum_{k \in \mathbb{Z}^d} \eta(n, i, k) \quad \text{and} \quad \eta(n) = \sum_{i \in Q_L} \eta(n, i).$$

Then $\eta(n)$ is the total number of particles, started in Q_L, and are surviving to time n. The quantity $E^{w_m}\eta(n)$ is the quenched expectation of the number of surviving particles. We'd like to understand the limiting behavior of $\frac{1}{B(n)} \sum_{j \in Q_{L(n)}} E^{w_m}\eta(n, j) - A(n)$ for various choices of L when these quantities tend to infinity as n does. In fact, this question is of interest in all of the models mentioned above and there are many others of this sort arising in applications of probability theory. This question already leads to interesting results in the case $\kappa = 0$, which we call the case of zero diffusivity, in analogy with terminology arising in the parabolic Anderson model. In this case

$$E^{w_m}\eta(n, i) = e^{-\beta \sum_{j=0}^{n} V_{ij}} \quad \text{and} \quad E^{w_m}\eta(n) = \sum_{i \in Q_L} e^{-\beta \sum_{j=0}^{n} V_{ij}}.$$

This is similar to the situation studied in Ben Arous, Bogachov Molchanov [1]. There, the authors considered a sequence of iid random variables $\{X_i\}_{i \geq 1}$

with subexponential tails. Various limit laws were derived for quantities of the form $\frac{1}{B(n)}\sum_{i=1}^{N(n)} e^{nX_i} - A(n)$ for various rates of growth of N. Briefly summarizing, in that work, a crucial quantity was $H(t) = \log E[e^{tX_i}]$. Then critical growth rates were determined in terms of H, call them $H_1(n)$ and $H_2(n)$, with $H_1(n) < H_2(n)$. For $N(n)$ growing faster than $H_1(n)$, a law of large numbers was established. For $N(n)$ growing faster than $H_2(n)$, a central limit theorem was established. For $N(n)$ growing more slowly than $H_2(n)$, stable limit laws were established.

This brings us to the topic dealt with here. The work of this paper should be viewed as the nonstationary analog of the results in [1]. In fact, taking $V_{ij} = X_i$ for each j gives the situation studied in that paper. Now, dropping the minus sign and specializing to the case $d = 1$, we study sums of the form

$$S_N(n) = \sum_{i=1}^{N} e^{\beta\sum_{j=1}^{n} V_{ij}}$$

where $\{V_{ij}, i, j \in \{1, 2, 3, \dots\}\}$ are non-negative *iid* random variables defined on some probability space $(\Omega, \mathcal{F}, P)$. Our goal is to understand how the growth rate of $N(n)$ influences the various limit laws that can arise for $S_N(n)$ when it is appropriately normalized and centered. Let V be a random variable, independent of and with the same law as the V_{ij}. Assume the Cramér condition

$$E[e^{sV}] = e^{\psi(s)} < \infty$$

for $s \in (-\infty, \beta_1)$ for some open interval containing the origin. It is useful to consider the properties of $\psi(s)$ for $s \in (-\infty, \beta_1)$. If we define Q_s by

$$\frac{dQ_s}{dP} = \frac{e^{sV}}{E[e^{sV}]},$$

then

$$\psi'(s) = \frac{E[Ve^{sV}]}{E[e^{sV}]} = E_{Q_s}[V],$$

and

$$\psi''(s) = \frac{E[V^2e^{sV}]}{E[e^{sV}]} - \left(\frac{E[Ve^{sV}]}{E[e^{sV}]}\right)^2 = \mathrm{Var}_{Q_s}(V) > 0.$$

Define

$$\lambda(s) = s\psi'(s) - \psi(s), \quad s \in (-\infty, \beta_1). \tag{1.2}$$

Note $\lambda'(s) = s\psi''(s)$, so λ is increasing on $(0, \beta_1)$. The definition of λ arises from the Legendre transform of ψ, which is given by

$$\psi^*(y) = \sup_{\lambda}\{\lambda y - \psi(y)\}. \tag{1.3}$$

A simple computation shows that given y, if there is an $s \in (-\infty, \beta_1)$ with $y = \psi'(s)$ then $\psi^*(y) = s\psi'(s) - \psi(s) = \lambda(s)$, but we shall not use this fact here.

From now on fix a β for which $\psi(2\beta) < \infty$, and define the two critical values,

$$\lambda_1 = \lambda(\beta) = \beta\psi'(\beta) - \psi(\beta) \quad \text{and} \quad \lambda_2 = \lambda(2\beta) = 2\beta\psi'(2\beta) - \psi(2\beta).$$

Since $\lambda(s)$ is an increasing function of s for $s > 0$, it follows that $\lambda_1 < \lambda_2$. Moreover, given any $\lambda \in (0, \lambda(\beta_1-))$, there is a unique $s \in (0, \beta_1)$, such that $\lambda = \lambda(s)$. We now state our results. Define

$$c_s(n) = s\sqrt{2\pi\psi''(s)n}, \quad 0 < s < \beta_1$$

and

$$N_s(n) = c_s(n)e^{\lambda(s)n}, \quad 0 < s < \beta_1.$$

We shall write $N(n)$ for the number of summands in the limit laws. When it appears as a subscript, as in S_N or $M_{N,k}$, the dependence on n will be suppressed. The law of large numbers and central limit theorem hold for S_N with $N(n) \geq e^{\lambda n}$ summands for λ above the critical values λ_1 and λ_2, respectively. The normalizations are different at these critical values than in the range $\lambda > \lambda_1$ for the law of large numbers and in the range $\lambda > \lambda_2$ for the central limit theorem. The normalizations at the critical parameters use

$$\tilde{A}_1(n) = E[e^{\beta\sum_{j=1}^n V_{ij}} 1_{\{\frac{1}{n}\sum_{j=1}^n V_{ij} \leq \psi'(\beta)\}}] \tag{1.4}$$

and

$$\tilde{A}_2(n) = E[e^{2\beta\sum_{j=1}^n V_{ij}} 1_{\{\frac{1}{n}\sum_{j=1}^n V_{ij} \leq \psi'(2\beta)\}}]. \tag{1.5}$$

THEOREM 1.1: *If $\lambda > \lambda_1$ and $\liminf_{n\to\infty} N(n)e^{-\lambda n} > 0$, then*

$$\lim_{n\to\infty} \frac{S_N(n)}{ES_N(n)} \stackrel{P}{=} 1. \tag{1.6}$$

For $N(n) = N_\beta(n)$,

$$\lim_{n\to\infty} \frac{S_N(n)}{N(n)\tilde{A}_1(n)} \stackrel{P}{=} 1. \tag{1.7}$$

THEOREM 1.2: *If $\lambda > \lambda_2$ and $\liminf_{n\to\infty} N(n)e^{-\lambda n} > 0$, then*

$$\lim_{n\to\infty} P\left(\frac{S_N(n) - ES_N(n)}{\sqrt{\operatorname{Var} S_N(n)}} < x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy. \tag{1.8}$$

If $N(n) = N_{2\beta}(n)$, then

$$\lim_{n\to\infty} P\left(\frac{S_N(n) - E[S_N(n)]}{\sqrt{N(n)\tilde{A}_2(n)}} \leq x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy. \tag{1.9}$$

If we consider $N(n) = N_s(n) = c_s(n)e^{\lambda(s)n}$ summands with s in the range $(0, 2\beta)$ we can obtain stable limit laws. This requires proper normalization and centering. Set

$$\alpha = s/\beta$$

and define the normalizing and centering constants

$$B_s(n) = e^{\beta\psi'(s)n} \tag{1.10}$$

and

$$A_\alpha(n) = \begin{cases} 0, & 0 < s < \beta, \\ \frac{N_\beta(n)}{B_s(n)} E[e^{\beta\sum_{j=1}^n V_{ij}} 1_{\{\frac{1}{n}\sum_{j=1}^n V_{ij} \leq \psi'(\beta)\}}], & s = \beta, \\ \frac{N_s(n)}{B_s(n)} E[e^{\beta\sum_{j=1}^n V_{ij}}], & \beta < s < 2\beta. \end{cases} \tag{1.11}$$

We note that

$$A_\alpha(n) = e^{(\lambda(s) - \beta\psi'(s) + \psi(\beta))n}, \quad \beta < s < 2\beta.$$

Then we have the following stable limit laws.

THEOREM 1.3: *If $0 < s < 2\beta$ and $N(n) = N_s(n) = c_s(n)e^{\lambda(s)n}$, then the limiting distribution of $B_s(n)^{-1}S_N(n) - A_\alpha(n)$ exists and has characteristic function*

$$f(u) = \exp\left\{i\gamma_\alpha u + \int_0^\infty \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)d\mathcal{L}_\alpha(x)\right\},$$

where

$$\mathcal{L}_\alpha(x) = \begin{cases} 0, & x < 0, \\ -x^{-\alpha}, & x > 0, \end{cases} \tag{1.12}$$

and

$$\gamma_\alpha = \begin{cases} \alpha\pi/2\cos(\alpha\pi/2) & \alpha \neq 1, \\ 0, & \alpha = 1. \end{cases} \tag{1.13}$$

We now turn to a consideration of order statistics. Define,

$$M_{N,1}(n) = \max\{e^{\beta\sum_{j=1}^n V_{ij}}; i \in \{1,\ldots,N(n)\}\}.$$

Then

$$M_{N,1}(n) = e^{\beta\sum_{j=1}^n V_{N^1 j}}$$

for some $N^1(n) \in \{1,\ldots,N(n)\}$. Set

$$M_{N,2}(n) = \max\{e^{\beta\sum_{j=1}^n V_{ij}}; i \in \{1,\ldots,N(n)\} \setminus \{N^1(n)\}\}$$

and let $N^2(n) \in \{1,\ldots,N(n)\}$ be such that

$$M_{N,2}(n) = e^{\beta\sum_{j=1}^n V_{N^2 j}}.$$

Continuing in this way, we obtain $M_{N,k}(n)$, $k \in \{1,\ldots,N(n)\}$ satisfying

$$M_{N,1}(n) \geq M_{N,2}(n) \geq \cdots \geq M_{N,N}(n).$$

Analogs of the classical limit laws of extreme value theory (see Leadbetter and Rootzen [4]) hold in the present context.

THEOREM 1.4: *For $0 < s < 2\beta$ and $\alpha = s/\beta, N(n) = N_s(n) = c_s(n)e^{\lambda(s)n}$,*

$$\lim_{n\to\infty} P\Big(\frac{M_{N,k}(n)}{B_s(n)} \leq x\Big) = \begin{cases} e^{-x^{-\alpha}}\sum_{j=0}^{k-1}\frac{x^{-j\alpha}}{j!}, & x > 0, \\ 0, & x \leq 0; \end{cases}$$

for $j < k$ and $x, y > 0$,

$$\lim_{n\to\infty} P\Big(\frac{M_{N,j}(n)}{B_s(n)} \leq x, \frac{M_{N,k}(n)}{B_s(n)} \leq y\Big)$$
$$= \begin{cases} e^{-y^{-\alpha}}\sum_{i=0}^{j-1}\sum_{\ell=0}^{k-1}\frac{x^{-j\alpha}(y^{-\alpha}-x^{-\alpha})}{j!(k-j)!}, & \text{if } x > y, \\ e^{-x^{-\alpha}}\sum_{i=0}^{j-1}\frac{x^{-j\alpha}}{j!}, & \text{if } x \leq y. \end{cases}$$

Remark: Specializing to the case $k = 1$ we have, with $N(n) = N_s(n)$, that

$$\frac{M_{N,1}(n)}{B_s(n)} = \frac{M_{N,1}(n)}{e^{\beta\psi(s)n}}$$

has the so-called Gumbel distribution as its limit law. Consequently,

$$\lim_{n\to\infty}\frac{\log M_{N,1}(n)}{\log B_s(n)} \stackrel{P}{=} 1.$$

Now for $0 < s < \beta$, from Theorem 1.3 we have

$$\lim_{n\to\infty}\frac{\log S_N(n)}{\log B_s(n)} \stackrel{P}{=} 1$$

so

$$\lim_{n\to\infty} \frac{\log M_{N,1}(n)}{\log S_N(n)} \overset{P}{=} 1.$$

Thus, on a logarithmic scale S_N and $M_{N,1}$ are comparable. That is, $S_N(n)$ is "dominated" by the largest summand. On the other hand, for values of $s \in (\beta, 2\beta)$ the law of large numbers (Theorem 1.1) holds for S_N when $N(n) = N_s(n) = c_s(n)e^{\lambda(s)}$, giving

$$\lim_{n\to\infty} \frac{\log S_N(n)}{\log(N(n)e^{n\psi(\beta)})} \overset{P}{=} 1.$$

Since Theorem 1.4 also holds for $N_s(n)$, we have

$$\lim_{n\to\infty} \frac{\log M_{N,1}(n)}{\log S_N(n)} \overset{P}{=} \frac{\beta\psi'(s)}{s\psi'(s) - \psi(s) + \psi(\beta)} < 1.$$

So for $\beta < s < 2\beta$, the maximum term does not account for the main growth of $S_N(n)$.

It's instructive to fix a value of $\lambda > 0$ and think of β as an inverse temperature which may vary. At high temperature, i.e. for β small, we'll have $\lambda > \lambda(2\beta)$ and 'disorder' is sufficient for the central limit theorem to hold. As temperature decreases, i.e. β increases, the central limit theorem ceases to hold once $\lambda(2\beta) > \lambda$, but the law of large numbers holds so long as $\lambda(\beta) > \lambda$. For temperatures in this last regime, $\lambda(\beta) > \lambda$, there is less 'disorder' and the maximum term begins to dominate in S_N.

The remainder of the paper is devoted to the proofs of these theorems.

2. Preliminary estimates

Recall the definition of $\lambda(s)$ at (1.2). It plays an important role in controlling

$$\mu_n[y, \infty) \equiv P\left(\frac{1}{n}\sum_{j=1}^{n} V_{ij} \geq y\right)$$

by means of error estimates in the central limit theorem applied to the Cramér transform of the distribution of V. Recall that $\lambda'(s) = s\psi''(s)$ and $\psi'(s) > 0$, $\psi''(s) > 0$ for $s \in (-\infty, \beta_1)$. Denote by μ the law of V and by μ_n the common law (over i) of $\frac{1}{n}\sum_{j=1}^{n} V_{ij}$. Given an $s \in (0, \beta_1)$, we define

$$\tilde{\mu}(dy) = \frac{e^{sy}}{e^{\psi(s)}}\mu(dy).$$

Denote by $\{\tilde{V}_j\}_{j\geq 1}$ an *iid* sequence of random variables with law $\tilde{\mu}$. Use $\tilde{\mu}_n$ to denote the law of their normalized sum:

$$\tilde{\mu}_n(dy) = P\Big(\frac{1}{n}\sum_{j=1}^{n} \tilde{V}_j \in dy\Big).$$

Notice that $E[\tilde{V}_j] = \psi'(s)$ and that $\mathrm{Var}(\tilde{V}_j) = \psi''(s)$. The value s will not appear in the notation for $\tilde{\mu}$ and $\tilde{\mu}_n$ as it always should be clear what it will be. Our first Lemma is a consequence of a common error bound in the central limit theorem. The relevant error bound appears as Theorem 5.22 from Petrov [6]. It says that if $EX_j = 0, EX_j^2 = \sigma^2, E|X_j|^3 < \infty$ with $\alpha = EX_1^3$ and

$$F_n(dx) = P\Big(\frac{1}{\sqrt{n\sigma^2}}\sum_{j=1}^{n} X_j \in dx\Big)$$

and $\Phi(x)$ is the distribution function for a $N(0,1)$ random variable and $\Psi(x) = 1 - \Phi(x)$, then uniformly in $x \in \mathbb{R}$,

$$1 - F_n(x) = \Psi(x) - \frac{\alpha}{6\sigma^3\sqrt{2\pi n}}(1-x^2)e^{-x^2/2} + o(n^{-1/2}). \tag{2.1}$$

This will give

LEMMA 2.1: *Given $0 < s < \beta_1$, there is a $c = c(s)$ such that for $z > 0$,*

$$\begin{aligned} e^{\lambda(s)n}\mu_n(z,\infty) = &\frac{1}{\sqrt{2\pi}}\int_{\sqrt{\frac{n}{\psi''(s)}}(z-\psi'(s))}^{\infty} e^{-s\sqrt{n\psi''(s)}v - v^2/2}dv \\ &+ \frac{c}{\sqrt{n}}\int_{\sqrt{\frac{n}{\psi''(s)}}(z-\psi'(s))}^{\infty} (v^3-3v)e^{-s\sqrt{n\psi''(s)}v-v^2/2}dv \\ &+ o(n^{-1/2})e^{-ns(z-\psi'(s))}. \end{aligned} \tag{2.2}$$

In addition, for a differentiable function f for which the integrals below converge and $a < b, a, b \in [-\infty, \infty]$, write

$$J = (a + \psi'(s), b + \psi'(s)) \quad \text{and} \quad I_n = \Big(\sqrt{\frac{n}{\psi''(s)}}a, \sqrt{\frac{n}{\psi''(s)}}b\Big).$$

Then,

$$
\begin{aligned}
&\int_J f(y)e^{\lambda(s)n}\mu_n(dy)\\
&=\frac{1}{\sqrt{2\pi}}\int_{I_n} f\left(\sqrt{\frac{\psi''(s)}{n}}v+\psi'(s)\right)e^{-s\sqrt{n\psi''(s)}v-v^2/2}dv\\
&\quad+\frac{c}{\sqrt{n}}\int_{I_n} f\left(\sqrt{\frac{\psi''(s)}{n}}v+\psi'(s)\right)(v^3-3v)e^{-s\sqrt{n\psi''(s)}v-v^2/2}dv\\
&\quad+o(n^{-1/2})e^{-ns(y-\psi'(s))}f(y)|_J-\int_J o(n^{-\frac{1}{2}})e^{-ns(y-\psi'(s))}f'(y)dy.
\end{aligned}
\tag{2.3}
$$

Proof: Notice that

$$
\begin{aligned}
\tilde{\mu}_n(y,\infty)&=P\left(\frac{1}{n}\sum_{j=1}^n\tilde{V}_j\ge y\right)\\
&=P\left(\frac{1}{\sqrt{n\psi''(s)}}\sum_{j=1}^n(\tilde{V}_j-\psi'(s))\ge\sqrt{\frac{n}{\psi''(s)}}(y-\psi'(s))\right),
\end{aligned}
$$

and we can apply (2.1) to the distribution of the quantity

$$\frac{1}{\sqrt{n\psi''(s)}}\sum_{j=1}^n(\tilde{V}_j-\psi'(s)).$$

By straightforward computation,

$$
\begin{aligned}
e^{\lambda(s)n}\mu_n(z,\infty)=&\int_z^\infty e^{-ns(y-\psi'(s))}\tilde{\mu}_n(dy)\\
=&-e^{-ns(y-\psi'(s))}\tilde{\mu}_n(y,\infty)|_z^\infty-ns\int_z^\infty e^{-ns(y-\psi'(s))}\tilde{\mu}_n(y,\infty)dy\\
=&-e^{-ns(y-\psi'(s))}\Psi(\sqrt{\frac{n}{\psi''(s)}}(y-\psi'(s))|_z^\infty\\
&-ns\int_z^\infty e^{-ns(y-\psi'(s))}\Psi(\sqrt{\frac{n}{\psi''(s)}}(y-\psi'(s))dy\\
&+\frac{c}{\sqrt{n}}e^{-ns(y-\psi'(s))}\left(1-\frac{n}{\psi''(s)}(y-\psi'(s))^2\right)e^{-\frac{n(y-\psi'(s))^2}{2\psi''(s)}}|_z^\infty\\
&+\frac{c}{\sqrt{n}}ns\int_z^\infty e^{-ns(y-\psi'(s))}\left(1-\frac{n}{\psi''(s)}(y-\psi'(s))^2\right)e^{-\frac{n(y-\psi'(s))^2}{2\psi''(s)}}dy\\
&+o(n^{-1/2})e^{-ns(y-\psi'(s))}|_z^\infty+o(n^{-1/2})ns\int_z^\infty e^{-ns(y-\psi'(s))}dy\\
=&\frac{1}{2\pi}\int_{\sqrt{\frac{n}{\psi''(s)}}(z-\psi'(s))}^\infty e^{-s\sqrt{n\psi''(s)}v-v^2/2}dv
\end{aligned}
$$

$$+\frac{c}{\sqrt{n}}\int_{\sqrt{\frac{n}{\psi''(s)}}(z-\psi'(s))}^{\infty}(v^3-3v)e^{-s\sqrt{n\psi''(s)}v-v^2/2}dv$$
$$+o(n^{-1/2})e^{-ns(z-\psi'(s))}.$$

The proof of (2.3) follows by first integrating by parts, applying (2.2) and then reversing the integration by parts in the terms not involving $o(n^{-1/2})$. ∎

COROLLARY 2.2: *With $0<s<2\beta$ and $\alpha=s/\beta$, we have*

$$\lim_{n\to\infty}N_s(n)\mu_n\Big(\frac{\log x}{\beta n}+\psi'(s),\infty\Big)=x^{-\alpha}.$$

Proof: Evaluating $e^{\lambda(s)n}\mu_n(z,\infty)$ at $z=(\log x)/\beta n+\psi'(s)$ and using Lemma 2.1, we have

$$\begin{aligned}N_s(n)\mu_n\Big(&\frac{\log x}{\beta n}+\psi'(s),\infty\Big)\\ =&c_s(n)\frac{1}{\sqrt{2\pi}}\int_{\frac{\log x}{\beta\sqrt{n\psi''(s)}}}^{\infty}e^{-s\sqrt{n\psi''(s)}v-v^2/2}dv\\ &+\frac{c}{\sqrt{n}}c_s(n)\int_{\frac{\log x}{\beta\sqrt{n\psi''(s)}}}^{\infty}(v^3-3v)e^{-s\sqrt{n\psi''(s)}v-v^2/2}dv+o(1).\end{aligned}$$

The first term becomes

$$\begin{aligned}c_s(n)\frac{1}{\sqrt{2\pi}}&\int_{\frac{\log x}{\beta\sqrt{n\psi''(s)}}}^{\infty}e^{-s\sqrt{n\psi''(s)}v-v^2/2}dv\\ &=s\sqrt{n\psi''(s)}\int_{\frac{\log x}{\beta\sqrt{n\psi''(s)}}}^{\infty}e^{-s\sqrt{n\psi''(s)}v-v^2/2}dv.\end{aligned}$$

But

$$\nu_n(dv)=s\sqrt{n\psi''(s)}e^{-s\sqrt{n\psi''(s)}v}1_{\{(\frac{\log x}{\beta\sqrt{n\psi''(s)}},\infty)\}}(v)dv$$

is a measure on $\mathbf{R}$ with total mass $\nu_n(\mathbf{R})=x^{-\alpha}$ which is converging weakly to $x^{-\alpha}\delta_0(dv)$ as $n\to\infty$. For the first term,

$$s\sqrt{n\psi''(s)}\int_{\frac{\log x}{\beta\sqrt{n\psi''(s)}}}^{\infty}e^{-s\sqrt{n\psi''(s)}v-v^2/2}dv=\int_{\mathbf{R}}e^{-v^2/2}\nu_n(dv)$$

and since $e^{-v^2/2}$ is a bounded continuous function, vanishing at infinity,

$$\lim_{n\to\infty}\int_{\mathbf{R}}e^{-v^2/2}\nu_n(dv)=x^{-\alpha}.$$

For the second term,

$$\frac{c}{\sqrt{n}}c_s(n)\int_{\frac{\log x}{\beta\sqrt{n\psi''(s)}}}^{\infty}(v^3-3v)e^{-s\sqrt{n\psi''(s)}v-v^2/2}dv$$
$$=\frac{c}{\sqrt{n}}\int_{\mathbf{R}}(v^3-3v)e^{-v^2/2}\nu_n(dv),$$

and so

$$\lim_{n\to\infty}\frac{c}{\sqrt{n}}\int_{\mathbf{R}}(v^3-3v)e^{-v^2/2}\nu_n(dv)=0.$$

Thus,

$$N_s(n)\mu_n\Big(\frac{\log x}{\beta n}+\psi'(s),\infty\Big)=\lim_{n\to\infty}\int_{\mathbf{R}}e^{-v^2/2}\nu_n(dv)=x^{-\alpha}$$

and that completes the proof. ∎

Another result we shall use later is

COROLLARY 2.3:

$$\lim_{n\longrightarrow\infty}e^{-\psi(2\beta)n}\tilde{A}_2(n)=\frac{1}{2}\quad\textit{and}\quad\lim_{n\longrightarrow\infty}c_{2\beta}(n)e^{-\psi(2\beta)n}\tilde{A}_2(n)=\infty.$$

Proof: We can apply (2.3) with $f(y)=e^{2\beta n(y-\psi'(2\beta))}$, $s=2\beta$ and $J=(0,\psi'(2\beta))$. This yields

$$e^{-\psi(2\beta)n}\tilde{A}_2(n)=\frac{1}{\sqrt{2\pi}}\int_{-\sqrt{\frac{n}{\psi''(2\beta)}}\psi'(2\beta)}^{0}e^{-v^2/2}dv$$
$$+\frac{c}{\sqrt{n}}\int_{-\sqrt{\frac{n}{\psi''(2\beta)}}\psi'(2\beta)}^{0}(v^3-2v)e^{-v^2/2}dv+o(1).$$

Both assertions of the Corollary now follow easily. ∎

3. Proofs of Theorems

Proof of Theorem 1.1: For (1.6), we assume $\liminf_{n\longrightarrow\infty}N(n)e^{-\lambda n}>0$, for some $\lambda>\lambda_1$. It is sufficient to show for some $\delta>0$,

$$\lim_{n\to\infty}\frac{E|S_N(n)-ES_N(n)|^{1+\delta}}{(ES_N(n))^{1+\delta}}=0. \tag{3.1}$$

First observe that by Jensen's inequality, since V is not identically constant,

$$e^{\psi((1+\delta)\beta)}=E[e^{(1+\delta)\beta V}]>(E[e^{\beta V}])^{1+\delta}=e^{(1+\delta)\psi(\beta)}.$$

Thus, by the von Bahr–Esseen inequality (exercise 2.6.20, page 82 in Petrov [6]), the numerator satisfies

$$\begin{aligned} E|S_N(n) - ES_N(n)|^{1+\delta} &\leq 2N(n)E|e^{\beta\sum_{j=1}^n V_{ij}} - e^{n\psi(\beta)}|^{1+\delta} \\ &\leq cN(n)(e^{n\psi((1+\delta)\beta)} + e^{n(1+\delta)\psi(\beta)}) \\ &\cong cN(n)e^{n\psi((1+\delta)\beta)}. \end{aligned}$$

On the other hand, the denominator satisfies

$$(ES_N(n))^{1+\delta} = e^{(1+\delta)n\psi(\beta)}N(n)^{(1+\delta)}.$$

By assumption, there is a $c > 0$ such that $N(n) \geq ce^{\lambda n}$ for n large enough. The ratio in (3.1) therefore eventually satisfies

$$\frac{E|S_N(n) - ES_N(n)|^{1+\delta}}{(ES_N(n))^{1+\delta}} \leq ce^{(-\delta\lambda+\psi((1+\delta)\beta)-(1+\delta)\psi(\beta))n}.$$

But

$$\psi((1+\delta)\beta) = \psi(\beta) + \psi'((1+\theta\delta)\beta)\beta\delta, \quad \text{for some } 0 < \theta < 1.$$

This implies, as $\lambda > \lambda_1$, that

$$-\delta\lambda + \psi((1+\delta)\beta) - (1+\delta)\psi(\beta) = \delta(\beta\psi'(\beta+\theta\delta\beta) - \psi(\beta) - \lambda) < 0$$

provided δ is chosen sufficiently small. Thus (3.1) holds and (1.6) is proved. We now prove (1.7), the law of large numbers at the critical value λ_1, that is for $N(n) = N_\beta(n)$. By Theorem 1.3, at $\lambda = \lambda_1 = \lambda(\beta)$, we have $\frac{1}{B_\beta(n)}S_N(n) - A_1(n)$ has a limiting distribution with $N(n) = N_\beta(n)$. Thus,

$$\frac{1}{A_1(n)}\Big(\frac{S_N(n)}{B_\beta(n)} - A_1(n)\Big) \xrightarrow{P} 0.$$

This is the same as

$$\frac{S_N(n)}{A_1(n)B_\beta(n)} \xrightarrow{P} 1.$$

Since $A_1(n)B_\beta(n) = N_\beta(n)\tilde{A}_1(n)$, this proves the result in the critical case $N(n) = N_\beta(n)$. ∎

We now turn to the proof of the central limit theorem.

Proof of Theorem 1.2: The proof of (1.8) uses the Lyapunov criteria, namely

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{N(n)} E|e^{\beta\Sigma_{j=1}^n V_{ij}} - (Ee^{\beta V})^n|^{2+\delta}}{(N(n)\operatorname{Var} e^{\beta\Sigma_{j=1}^n V_{ij}})^{1+\delta/2}} = 0, \quad \text{for some } \delta > 0, \tag{3.2}$$

implies the central limit theorem. Using the assumption $\liminf_{n\to\infty} e^{-\lambda n}N(n) > 0$, the ratio in the limit in (3.2) is eventually bounded above as follows:

$$\begin{aligned}\frac{\sum_{i=1}^{N(n)} E|e^{\beta\Sigma_{j=1}^n V_{ij}} - (Ee^{\beta V})^n|^{2+\delta}}{(N(n)\operatorname{Var} e^{\beta\Sigma_{j=1}^n V_{ij}})^{1+\delta/2}} &= N(n)^{-\delta/2}e^{(\psi((2+\delta)\beta)-(1+\delta/2)\psi(2\beta))n}\\ &\le ce^{-n(\frac{\delta}{2}\lambda-\psi((2+\delta)\beta)+(1+\delta/2)\psi(2\beta))}\\ &= ce^{-n(\frac{\delta}{2}\lambda-\psi((1+\delta/2)2\beta)+(1+\delta/2)\psi(2\beta))}\\ &= ce^{-n(\lambda-2\beta\psi'(2\beta+\theta\delta\beta)+\psi(2\beta))\frac{\delta}{2}},\\ &\quad\text{for some } \theta\in(0,1).\end{aligned}$$

Then, provided δ is sufficiently small and $\lambda > \lambda_2$ we have

$$\lambda - 2\beta\psi'(2\beta+\theta\delta\beta) + \psi(2\beta) > 0.$$

The limit in (3.2) is thus 0 and (1.8) is proved. For the proof of (1.9), define

$$Y(n) = \frac{e^{\beta\sum_{j=1}^n V_{ij}}}{\sqrt{N_{2\beta}(n)\tilde{A}_2(n)}}.$$

It suffices to show the following three conditions hold for every $\tau > 0$:

$$\begin{gathered}\text{(a)} \lim_{n\to\infty} N_{2\beta}(n)P(Y(n)>\tau) = 0,\\ \text{(b)} \lim_{n\to\infty} N_{2\beta}(n)E[Y(n)1_{\{Y(n)>\tau\}}] = 0,\\ \text{(c)} \lim_{n\to\infty} N(n)(E[Y^2(n)1_{\{Y(n)\le\tau\}}] - (E[Y(n)1_{\{Y(n)\le\tau\}}])^2) = 1.\end{gathered}$$

For (a), using Lemma 2.1 and writing

$$f_n = c_{2\beta}(n)e^{-\psi(2\beta n)}\tilde{A}_2(n),$$

and

$$d_n = \frac{\log(\tau\sqrt{f_n})}{\beta n} \quad\text{and}\quad e_n = \sqrt{\frac{n}{\psi''(2\beta)}}d_n,$$

we obtain

$$\begin{aligned}N_{2\beta}(n)P(Y(n)\ge\tau) =& c_{2\beta}(n)e^{\lambda(2\beta)n}\mu_n\Big(\frac{\log\tau}{\beta n} + \frac{\log(N_{2\beta}(n)\tilde{A}_2(n))}{2\beta n},\infty\Big)\\ =& c_{2\beta}(n)e^{\lambda(2\beta)n}\mu_n(d_n+\psi'(2\beta),\infty)\\ =& 2\beta\sqrt{n\psi''(2\beta)}\int_{e_n}^\infty e^{-2\beta\sqrt{n\psi''(2\beta)}v-v^2/2}dv\\ &+\frac{c}{\sqrt{n}}\sqrt{n\psi''(2\beta)}\int_{e_n}^\infty (v^3-3v)e^{-2\beta\sqrt{n\psi''(2\beta)}v-v^2/2}dv\\ &+o(1).\end{aligned}$$

Notice that

$$\gamma_n(dy) = 2\beta\sqrt{n\psi''(2\beta)}e^{-2\beta\sqrt{n\psi''(2\beta)}v}1_{(e_n,\infty)}(v)dv$$

is a measure on $\mathbf{R}$ with total mass $(\tau\sqrt{f_n})^{-2}$ which, by Corollary 2.3, tends to 0 as $n \longrightarrow \infty$. Thus, the last three terms vanish as $n \longrightarrow \infty$ and (a) holds. For the proof of (b), we see by (2.3) that

$$\begin{aligned}
&N_{2\beta}(n)E[Y(n)1_{\{Y(n)>\tau\}}] \\
&\quad = \frac{c_{2\beta}(n)}{\sqrt{f_n}}\int_{d_n}^{\infty} e^{\beta n(y-\psi'(2\beta))}e^{\lambda(2\beta)n}\mu_n(dy) \\
&\quad = 2\beta\sqrt{\frac{n\psi''(2\beta))}{f_n}}\int_{e_n}^{\infty} e^{-\beta\sqrt{n\psi''(2\beta))}v-v^2/2}dv \\
&\quad\quad + \frac{c}{\sqrt{n}}\sqrt{\frac{n\psi''(2\beta))}{f_n}}\int_{e_n}^{\infty} e^{-\beta\sqrt{n\psi''(2\beta))}v-v^2/2}(v^3-2v)dv \\
&\quad\quad + \frac{o(1)}{\tau f_n}.
\end{aligned}$$

Noticing that

$$\zeta_n(dv) = 2\beta\sqrt{n\psi''(2\beta)}1_{(e_n,\infty)}(v)e^{-\beta\sqrt{n\psi''(2\beta))}v}dv$$

is a measure on $\mathbf{R}$ with total mass $\zeta_n(\mathbf{R}) = 2(\tau\sqrt{f_n})^{-1}$, and since $\lim_{n\longrightarrow\infty} f_n = \infty$, it follows that ζ_n is converging weakly to the zero measure. Thus,

$$\lim_{n\longrightarrow\infty} N_{2\beta}(n)E[Y(n)1_{\{Y(n)>\tau\}}] = 0.$$

We now turn to the proof of (c). First observe that

$$\begin{aligned}
\frac{(E[Y(n)1_{\{Y(n)\le\tau\}}])^2}{E[Y^2(n)1_{\{Y(n)\le\tau\}}]} &\le \frac{e^{2\psi(\beta)n}}{E[e^{2\beta\sum_1^n V_{ij}}1_{\{\frac{1}{t}\sum_1^n V_{ij}\le\psi'(2\beta)\}}]} \\
&= \frac{e^{2\psi(\beta)n}}{\tilde{A}_2(n)} \\
&= \frac{e^{(2\psi(\beta)-\psi(2\beta))n}}{e^{-\psi(2\beta)n}\tilde{A}_2(n)}.
\end{aligned}$$

By Jensen's inequality,

$$e^{2\psi(\beta)n} < e^{\psi(2\beta))n}$$

which implies, using Corollary 2.3, that

$$\lim_{n\to\infty} \frac{(E[Y(n)1_{\{Y(n)\le\tau\}}])^2}{E[Y^2(n)1_{\{Y(n)\le\tau\}}]} = 0.$$

This reduces (c) to showing that

$$\lim_{n\to\infty} N_{2\beta}(n)E[Y^2(n)1_{\{Y(n)\le\tau\}}] = 1.$$

But, by Lemma 2.1,

$$\begin{aligned}
&N_{2\beta}(n)E[Y^2(n)1_{\{Y(n)\le\tau\}}]\\
&\quad=\frac{c_{2\beta}(n)}{f_n}\int_0^{d_n+\psi'(2\beta)} e^{2\beta n(y-\psi'(2\beta))}e^{\lambda(2\beta)n}\mu_n(dy)\\
&\quad=\frac{1}{e^{-\psi(2\beta)n}\tilde{A}_2(n)\sqrt{2\pi}}\int_{-\sqrt{\frac{n}{\psi''(2\beta)}}\psi'(2\beta)}^{e_n} e^{-v^2/2}dv\\
&\quad\quad+\frac{c}{\sqrt{n}e^{-\psi(2\beta)n}\tilde{A}_2(n)}\int_{-\sqrt{\frac{n}{\psi''(2\beta)}}\psi'(2\beta)}^{e_n}(v^3-2v)e^{-v^2/2}dv\\
&\quad\quad+o(1).
\end{aligned}$$

In the first term, $\lim_{n\longrightarrow\infty} e_n = 0$, $\lim_{n\longrightarrow\infty}\sqrt{\frac{n}{\psi''(2\beta)}}\psi'(2\beta) = \infty$, and by Corollary 2.3, we have $\lim_{n\longrightarrow\infty} e^{-\psi(2\beta)n}\tilde{A}_2(n) = \frac{1}{2}$, so

$$\lim_{n\longrightarrow\infty}\frac{1}{e^{-\psi(2\beta)n}\tilde{A}_2(n)\sqrt{2\pi}}\int_{-\sqrt{\frac{n}{\psi''(2\beta)}}\psi'(2\beta)}^{e_n} e^{-v^2/2}dv = 1.$$

The remaining terms are easily seen to tend to zero as $n \longrightarrow \infty$. This concludes the proof of (c) and therefore Theorem 1.2 is proved. ∎

Proof of Theorem 1.3: The proof of Theorem 1.3 requires verification (via an appropriate modification of Theorem 3.4 from Petrov [6]) of the following three items:

(1)

$$\begin{aligned}
\lim_{n\to\infty} N_s(n)P(B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}} \le x) &= \mathcal{L}_\alpha(x), \quad x<0,\\
-\lim_{n\to\infty} N_s(n)P(B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}} > x) &= \mathcal{L}_\alpha(x), \quad x>0.
\end{aligned}$$

(2)

$$\sigma^2 = \lim_{\tau\searrow 0}\overline{\lim_{n\to\infty}} N_s(n)\operatorname{Var}\left(B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}}1_{\{B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}}\le\tau\}}\right) = 0.$$

(3) There is a γ such that for each $\tau > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} N_s(n)E\left(B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}}1_{\{B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}}\le\tau\}}\right) - A_\alpha(n)\\
&\quad=\gamma+\int_0^\tau\frac{x^3}{1+x^2}d\mathcal{L}_\alpha(x)-\int_\tau^\infty\frac{x}{1+x^2}d\mathcal{L}_\alpha(x).
\end{aligned}$$

Proof of (1): Let $0 < s < 2\beta$. Obviously, $P(B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}} \leq x) = 0$ for $x < 0$, so there is nothing to prove in this case. For $x > 0$, by Corollary 2.2, we have

$$\lim_{n\to\infty} N_s(n)P(B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}} > x) = \lim_{n\to\infty} N_s(n)\mu_n\Big(\frac{\log x}{\beta n} + \psi'(s), \infty\Big) = x^{-\alpha},$$

so (1) is established. ∎

Proof of (2): For $0 < s < 2\beta$, notice that

$$\xi_n(dv) = s\sqrt{n\psi''(s)}e^{(2\beta-s)\sqrt{n\psi''(s)}v}1_{(-\sqrt{\frac{n}{\psi''(s)}}\psi'(s), \frac{\log\tau}{\beta\sqrt{n\psi''(s)}})}(v)dv$$

has total mass $\frac{\alpha}{2-\alpha}\tau^{2-\alpha}(1+o(1))$ and converges weakly to $\frac{\alpha}{2-\alpha}\tau^{2-\alpha}\delta_0(dv)$ as $n \longrightarrow \infty$. Writing

$$\eta_\tau(n) = \psi'(s) + \frac{\log\tau}{\beta n},$$

and using Lemma 2.1,

$$\begin{aligned}
&N_s(n)\operatorname{Var}\big(B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}}1_{\{B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}}\leq\tau\}}\big)\\
&\leq\frac{N_s(n)}{B_s(n)^2}E[e^{2\beta\sum_{j=1}^n V_{ij}}1_{\{\frac{1}{n}\sum_{j=1}^n V_{ij}\leq\eta_\tau(n)\}}]\\
&=c_s(n)\int_0^{\eta_\tau(n)} e^{2\beta n(y-\psi'(s))}e^{\lambda(s)n}\mu_n(dy)\\
&=\int_{-\infty}^{\infty} e^{-v^2/2}\xi_n(dv)\\
&\quad+\frac{c}{\sqrt{n}}\int_{-\infty}^{\infty}(v^3-3v)e^{-v^2/2}\xi_n(dv)\\
&\quad+o(1)e^{(2\beta-s)n(y-\psi'(s))}\big|_0^{\eta_\tau(n)}\\
&\quad+o(1)\beta n\int_0^{\eta_\tau(n)} e^{(2\beta-s)n(y-\psi'(s))}dy.
\end{aligned}$$

The first term satisfies

$$\lim_{n\longrightarrow\infty}\int_{-\infty}^{\infty} e^{-v^2/2}\xi_n(dv) = \frac{\alpha}{2-\alpha}\tau^{2-\alpha},$$

while the second satisfies

$$\lim_{n\longrightarrow\infty}\frac{c}{\sqrt{n}}\int_{-\infty}^{\infty}(v^3-3v)e^{-v^2/2}\xi_n(dv) = 0$$

and the third satisfies

$$\lim_{n\longrightarrow\infty} o(1)e^{(2\beta-s)n(y-\psi'(s))}\big|_0^{\eta_\tau(n)} + o(1)\beta n\int_0^{\eta_\tau(n)} e^{(2\beta-s)n(y-\psi'(s))}dy = 0.$$

Thus

$$\lim_{n\longrightarrow\infty} N_s(n)\operatorname{Var}\left(B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}}1_{\{B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}}\leq\tau\}}\right) = \frac{\alpha}{2-\alpha}\tau^{2-\alpha}.$$

Since $\lim_{\tau\searrow 0}\frac{\alpha}{2-\alpha}\tau^{2-\alpha} = 0$, this implies (2). ∎

Proof of (3): We first observe that $d\mathcal{L}_\alpha(x) = \alpha x^{-1-\alpha}dx$ and compute the right hand side of (3), which is

$$I(\alpha) \equiv \gamma + \alpha\int_0^\tau \frac{x^3}{1+x^2}x^{-1-\alpha}dx - \alpha\int_\tau^\infty \frac{x}{1+x^2}x^{-1-\alpha}dx.$$

Thus, for $0<\alpha<1$,

$$\begin{aligned} I(\alpha) =& \gamma + \frac{\alpha}{1-\alpha}\tau^{1-\alpha} - \alpha\int_0^\infty \frac{x^{-\alpha}}{1+x^2}dx \\ =& \gamma + \frac{\alpha}{1-\alpha}\tau^{1-\alpha} - \frac{\alpha\pi}{2\cos(\alpha\pi/2)}. \end{aligned}$$

For $1<\alpha<2$,

$$\begin{aligned} I(\alpha) =& \gamma + \frac{\alpha}{1-\alpha}\tau^{1-\alpha} + \alpha\int_0^\infty \frac{x^{2-\alpha}}{1+x^2}dx \\ =& \gamma + \frac{\alpha}{1-\alpha}\tau^{1-\alpha} - \frac{\alpha\pi}{2\cos(\alpha\pi/2)}. \end{aligned}$$

Finally, for $\alpha=1$,

$$\begin{aligned} I(\alpha) =& \gamma + \int_0^\tau \frac{x}{1+x^2}dx - \int_\tau^\infty \frac{x^{-1}}{1+x^2}dx \\ =& \gamma + \frac{1}{2}\log(1+\tau^2) - \left(\frac{1}{2}\log(1+\tau^2) - \log\tau\right) \\ =& \gamma + \log\tau. \end{aligned}$$

We'll show that for $0<\alpha<1$, or $1<\alpha<2$,

$$\gamma = \frac{\alpha\pi}{2\cos(\alpha\pi/2)},$$

while for $\alpha = 1$,

$$\gamma = 0.$$

This will be accomplished by showing that the limit in (3) is $\frac{\alpha}{1-\alpha}\tau^{1-\alpha}$ for $\alpha\neq 1$ and is $\log\tau$ when $\alpha = 1$.

CASE (i): $0 < s < \beta$.

Then $A_\alpha(n) \equiv 0$, $B_s(n) = e^{\beta\psi'(s)n}$, $N_s(n) = c_s(n)e^{\lambda(s)n}$, and recalling the notation

$$\eta_\tau(n) = \psi'(s) + \frac{\log\tau}{\beta n},$$

using Lemma 2.1 we have

$$\begin{aligned}
&\frac{N_s(n)}{B_s(n)}E[e^{\beta\sum_{j=1}^n V_{ij}}1_{\{\frac{1}{B_s(n)}e^{\beta\sum_{j=1}^n V_{ij}}\le\tau\}}]\\
=&c_s(n)\int_0^{\eta_\tau(n)} e^{\beta n(y-\psi'(s))}e^{\lambda(s)n}\mu_n(dy)\\
=&s\sqrt{n\psi''(s)}\int_{-\sqrt{\frac{n}{\psi''(s)}}\psi'(s)}^{\frac{\log\tau}{\beta\sqrt{n\psi''(s)}}} e^{(\beta-s)\sqrt{n\psi''(s)}v-v^2/2}dv\\
&+cs\sqrt{\psi''(s)}\int_{-\sqrt{\frac{n}{\psi''(s)}}\psi'(s)}^{\frac{\log\tau}{\beta\sqrt{n\psi''(s)}}}(v^3-3v)e^{(\beta-s)\sqrt{n\psi''(s)}v-v^2/2}dv\\
&+o(1)e^{(\beta-s)n(y-\psi'(s))}|_0^{\eta_\tau(n)}\\
&+o(1)\beta n\int_0^{\eta_\tau(n)} e^{(\beta-s)n(y-\psi'(s))}dy.
\end{aligned}$$

For the first term, notice that

$$\chi_n(dv) = s\sqrt{n\psi''(s)}e^{(\beta-s)\sqrt{n\psi''(s)}v}1_{\{(-\sqrt{\frac{n}{\psi''(s)}}\psi'(s),\frac{\log\tau}{\beta\sqrt{n\psi''(s)}})\}}(v)dv$$

is a measure on $\mathbf{R}$ with total mass $\frac{\alpha}{1-\alpha}\tau^{1-\alpha}(1+o(1))$. Moreover, $\chi_n(dv)$ is converging weakly to $\frac{\alpha}{1-\alpha}\tau^{1-\alpha}\delta_0(dv)$. Thus,

$$\lim_{n\longrightarrow\infty} s\sqrt{n\psi''(s)}\int_{-\sqrt{\frac{n}{\psi''(s)}}\psi'(s)}^{\frac{\log\tau}{\beta\sqrt{n\psi''(s)}}} e^{(\beta-s)\sqrt{n\psi''(s)}v-v^2/2}dv = \frac{\alpha}{1-\alpha}\tau^{1-\alpha}.$$

The same observation implies that

$$\lim_{n\longrightarrow\infty} cs\sqrt{\psi''(s)}\int_{-\sqrt{\frac{n}{\psi''(s)}}\psi'(s)}^{\frac{\log\tau}{\beta\sqrt{n\psi''(s)}}}(v^3-3v)e^{(\beta-s)\sqrt{n\psi''(s)}v-v^2/2}dv = 0.$$

Finally, it is easy to see that

$$\lim_{n\longrightarrow\infty} o(1)e^{(\beta-s)n(y-\psi'(s))}|_0^{\eta_\tau(n)} + o(1)\beta n\int_0^{\eta_\tau(n)} e^{(\beta-s)n(y-\psi'(s))}dy = 0.$$

Thus,

$$\lim_{n\longrightarrow\infty}\frac{N_s(n)}{B_s(n)}E[e^{\beta\sum_{j=1}^n V_{ij}}1_{\{\frac{1}{B_s(n)}e^{\beta\sum_{j=1}^n V_{ij}}\le\tau\}}] = \frac{\alpha}{1-\alpha}\tau^{1-\alpha}$$

and the proof of (3) for $0 < s < \beta$ is complete.

CASE (ii): $s = \beta$ $(\alpha = 1)$.

In this case, $A_1(n) = \frac{N_\beta(n)}{B_\beta(n)} E[e^{\beta \sum_{j=1}^n V_{ij}} 1_{\{\frac{1}{n}\sum_{j=1}^n V_{ij} \le \psi'(\beta)\}}]$ and we must prove

$$\lim_{n \longrightarrow \infty} \frac{N_\beta(n)}{B_\beta(n)} E[e^{\beta \sum_{j=1}^n V_{ij}} 1_{\{\frac{1}{B_\beta(n)} e^{\beta \sum_{j=1}^n V_{ij}} \le \tau\}}] - A_1(n) = \log \tau.$$

Proceeding in a manner similar to case (i), we use Lemma 2.1 to get

$$\begin{aligned}
&\frac{N_\beta(n)}{B_\beta(n)} E[e^{\beta \sum_{j=1}^n V_{ij}} 1_{\{\frac{1}{B_\beta(n)} e^{\beta \sum_{j=1}^n V_{ij}} \le \tau\}}] - A_1(n) \\
&= \frac{N_\beta(n)}{B_\beta(n)} E[e^{\beta \sum_{j=1}^n V_{ij}} 1_{\{\psi'(\beta) \le \frac{1}{n}\sum_{j=1}^n V_{ij} \le \eta_\tau(n)\}}] \\
&= c_\beta(n) \int_{\psi'(\beta)}^{\eta_\tau(n)} e^{\beta n(y - \psi'(\beta))} e^{\lambda(\beta) n} \mu_n(dy) \\
&= \beta \sqrt{n \psi''(\beta)} \int_0^{\frac{\log \tau}{\beta \sqrt{n \psi''(s)}}} e^{-v^2/2} dv \\
&\quad + c\beta \sqrt{\psi''(\beta)} \int_0^{\frac{\log \tau}{\beta \sqrt{n \psi''(s)}}} (v^3 - 3v) e^{-v^2/2} dv \\
&\quad + o(1).
\end{aligned}$$

Now the first term satisfies

$$\lim_{n \longrightarrow \infty} \beta \sqrt{n \psi''(\beta)} \int_0^{\frac{\log \tau}{\beta \sqrt{n \psi''(s)}}} e^{-v^2/2} dv = \log \tau,$$

while $c\beta \sqrt{\psi''(\beta)} \int_0^{\frac{\log \tau}{\beta \sqrt{n \psi''(s)}}} (v^3 - 3v) e^{-v^2/2} dv$ vanishes in the limit as $n \longrightarrow \infty$. That finishes the proof of the case $s = \beta$.

CASE (iii): $\beta < s < 2\beta$.

Here, we must show

$$\lim_{n \longrightarrow \infty} \frac{N_s(n)}{B_s(n)} E[e^{\beta \sum_{j=1}^n V_{ij}} 1_{\{\frac{1}{B_s(n)} e^{\beta \sum_{j=1}^n V_{ij}} \le \tau\}}] - A_\alpha(n) = \frac{\alpha}{1 - \alpha} \tau^{1-\alpha}.$$

But, by the definition of $A_\alpha(n)$ and Lemma 2.1 we have

$$\begin{aligned}
&N_s(n)E[B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}}1_{\{B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}}\le\tau\}}]-A_\alpha(n)\\
=&N_s(n)E[B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}}1_{\{B_s(n)^{-1}e^{\beta\sum_{j=1}^n V_{ij}}>\tau\}}]\\
=&s\sqrt{n\psi''(s)}\int_{\frac{\log\tau}{\beta\sqrt{n\psi''(s)}}}^{\infty}e^{(\beta-s)\sqrt{n\psi''(s)}v-v^2/2}dv\\
&+\frac{c}{\sqrt{n}}s\sqrt{n\psi''(s)}\int_{\frac{\log\tau}{\beta\sqrt{n\psi''(s)}}}^{\infty}(v^3-3v)e^{(\beta-s)\sqrt{n\psi''(s)}v-v^2/2}dv\\
&+o(1).
\end{aligned}$$

The measure

$$\pi_n(dv)=s\sqrt{n\psi''(s)}1_{\{(\frac{\log\tau}{\beta\sqrt{n\psi''(s)}},\infty)\}}e^{(\beta-s)\sqrt{n\psi''(s)}v}dv$$

has total mass $\frac{\alpha}{1-\alpha}\tau^{1-\alpha}(1+o(1))$ and is converging weakly to $\frac{\alpha}{1-\alpha}\tau^{1-\alpha}\delta_0(dv)$. Thus,

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}s\sqrt{n\psi''(s)}\int_{\frac{\log\tau}{\beta\sqrt{n\psi''(s)}}}^{\infty}e^{(\beta-s)\sqrt{n\psi''(s)}v-v^2/2}dv=\frac{\alpha}{1-\alpha}\tau^{1-\alpha},\\
&\lim_{n\longrightarrow\infty}\frac{c}{\sqrt{n}}s\sqrt{n\psi''(s)}\int_{\frac{\log\tau}{\beta\sqrt{n\psi''(s)}}}^{\infty}(v^3-2v)e^{(\beta-s)\sqrt{n\psi''(s)}v-v^2/2}dv=0
\end{aligned}$$

and thus

$$\lim_{n\longrightarrow\infty}\frac{N_s(n)}{B_s(n)}E[e^{\beta\sum_{j=1}^n V_{ij}}1_{\{\frac{1}{B_s(n)}e^{\beta\sum_{j=1}^n V_{ij}}\le\tau\}}]-A_\alpha(n)=\frac{\alpha}{1-\alpha}\tau^{1-\alpha},$$

as desired. ∎

This competes the proof of Theorem 1.3. ∎

Proof of Theorem 1.4: We only give the proof for the limiting distribution of $M_{N,1}/B_s(n)$; the other proofs follow in a similar manner. Recall

$$\begin{aligned}
M_{N,1}&=\max\{e^{\beta\sum_{j=1}^n V_{ij}},\quad i=1,2,\dots,N(n)\},\\
B_s(n)&=e^{\beta\psi'(s)n},\quad 0<s<2\beta,\\
N_s(n)&=c_s(n)e^{\lambda(s)n},
\end{aligned}$$

and we want to prove that if $N(n)=N_s(n)$, then

$$P\Big(\frac{M_{N,1}}{B_s(n)}\le x\Big)=\begin{cases}e^{-x^{-\alpha}}, & x>0,\\ 0, & \text{otherwise.}\end{cases}$$

Now, for $x > 0$,

$$\begin{aligned} P(M_{N,1} \leq xB_s(n)) &= P(e^{\beta \sum_{j=1}^{n} V_{ij}} \leq xB_s(n))^{N_s(n)} \\ &= P\Big(\frac{1}{n}\sum_{j=1}^{n} V_{ij} \leq \frac{\log x}{\beta n} + \psi'(s)\Big)^{N_s(n)} \\ &= \Big(1 - \mu_n(\psi'(s) + \frac{\log x}{\beta n}, \infty)\Big)^{N_s(n)}. \end{aligned}$$

By Corollary 2.2, if $x > 0$,

$$N_s(n)\mu_n\Big(\psi'(s) + \frac{\log x}{\beta n}, \infty\Big) = x^{-\alpha}(1 + o(1)).$$

Therefore, for $x > 0$,

$$\begin{aligned} \Big(1 - \mu_n(\psi'(s) + \frac{\log x}{\beta n}, \infty)\Big)^{N_s(n)} &= \Big(1 - \frac{N_s(n)\mu_n(\psi'(s) + \frac{\log x}{\beta n}, \infty)}{N_s(n)}\Big)^{N_s(n)} \\ &= \Big(1 - \frac{x^{-\alpha}(1 + o(1))}{N_s(n)}\Big)^{N_s(n)} \\ &\longrightarrow e^{-x^{-\alpha}}, \quad n \longrightarrow \infty. \end{aligned}$$

For $x \leq 0$ there is nothing to prove. ∎

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